Isomorphic Semigroups of Boolean Matrices

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Definitions

Semigroup: A set with the properties of a group but lacking identities and/or inverses

- Positive integers under addition
- Integers under multiplication

More definitions…

Isomorphism: a one-to-one and onto correspondence between two mathematical sets.

- ‘One, two three…’ and ‘Uno, dos, tres…’

3x3 boolean matrices

$$A = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$B = \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

$$B^2 = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad AB = \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad BA = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}$$

Cayley table for A and B

	A	B	B ^ 2	A B	B A
A	A	AB	B^2	AB	B^2
B	BA	B^2	B^2	B^2	B^2
B ^ 2	B^2	B^2	B^2	B^2	B^2
A B	B^2	B^2	B^2	B^2	B^2
B A	BA	B^2	B^2	B^2	B^2

More matrices

C =

1	0	0
1	1	0
1	1	1

D =

0	1	1
1	1	1
1	1	0

	C	**D**	**D^2**	**CD**	**DC**
C	C	CD	D^2	DC	D^2
D	DC	D^2	D^2	D^2	D^2
D^2	D^2	D^2	D^2	D^2	D^2
CD	D^2	D^2	D^2	D^2	D^2
DC	DC	D^2	D^2	D^2	D^2

	A	B	B^2	AB	BA
A	A	AB	B^2	AB	B^2
B	BA	B^2	B^2	B^2	B^2
B^2	B^2	B^2	B^2	B^2	B^2
AB	B^2	B^2	B^2	B^2	B^2
BA	BA	B^2	B^2	B^2	B^2

	C	D	D^2	CD	DC
C	C	CD	D^2	CD	D^2
D	DC	D^2	D^2	D^2	D^2
D^2	D^2	D^2	D^2	D^2	D^2
CD	D^2	D^2	D^2	D^2	D^2
DC	DC	D^2	D^2	D^2	D^2

What is the same about these semigroups?

They’re isomorphic!

- Elements generated by A and B have the same role as those generated by C and D
- Each element in one semigroup has a corresponding element in the other

This is just one example of how isomorphism can relate to semigroups of matrices. So, we looked for other ways these concepts could be related…

Even more definitions

(Note: not a generally accepted definition – we invented it)

- Order of a matrix: For matrix M of order n,

  $M^n = M^{n+1}$ (where n is minimum).

1	1	1
1	1	1
1	1	1

→

1	1	1
1	1	1
1	1	1

Order 1

0	0	1
1	0	1
1	1	0

→

1	1	1
1	1	1
1	1	1

→

1	1	1
1	1	1
1	1	1

Order 2

Our theorem

- A matrix of order n generates a semigroup with exactly n elements.
- The semigroups generated by all matrices of order n are isomorphic to each other.

Proof: Take matrix M of order n. Set m is defined as {M, M2,…, M_n}. This set will have n elements. Multiplication of matrices is associative, so the set is associative under multiplication. Any M_i x M_j, i,j>0 (that is, any two members of the set multiplied) will give an already existing member of the set; if i+j<n then M_{i+j} is already in the set, and if i+j>n then M_{i+j} = M_n. So set m is also closed, and therefore is a semigroup. Matrix N will generate the same set with N instead of M, so the two sets are isomorphic.

Conjecture

- We conjectured that if two 3x3 matrices are of order 1, their product is also of order 1. More generally, if 3x3 matrices A, B of orders c, d are multiplied, their product would be of order ≤ cd. We haven't found any counterexample so far, but we haven't proved it either…